

# Vortex Solutions of Maxwell-Chern-Simons field coupled to 4-Fermion theory

Seungjoon Hyun, Junsoo Shin and Jae Hyung Yee

*Department of Physics and Institute for Mathematical Sciences  
Yonsei University  
Seoul 120-749, Korea*

Hyuk-jae Lee

*Department of Physics  
Boston University  
Boston, Massachusetts 02215, U.S.A.*

## Abstract

We find the static vortex solutions of the model of Maxwell-Chern-Simons gauge field coupled to a (2+1)-dimensional four-fermion theory. Especially, we introduce two matter currents coupled to the gauge field minimally: the electromagnetic current and a topological current associated with the electromagnetic current. Unlike other Chern-Simons solitons the N-soliton solution of this theory has binding energy and the stability of the solutions is maintained by the charge conservation laws.

Various field theories which include Chern-Simons terms in (2+1)-dimensions are found to admit interesting classical soliton solutions [1][2][3][4][5]. Recently it has been found that fermionic field theories coupled to the Chern-Simons gauge field also admit vortex solutions [4][5]. In these models the fermionic fields are coupled to Chern-Simons gauge field without Maxwell term. In this note we present a fermion field theory coupled to Maxwell-Chern-Simons field which possesses interesting vortex solutions with binding energies.

We consider a four-fermion model coupled to Maxwell-Chern-Simons theory that interacts electromagnetically in (2+1) dimensions as in Ref.[6]. The four-fermion models in (2+1) dimensions are non-renormalizable in the weak coupling expansion, but is known to be renormalizable in  $\frac{1}{N}$  expansion [7],  $N$  being the number of flavours.

We consider the model described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + i\bar{\psi}^a\gamma^\mu\partial_\mu\psi^a - m\bar{\psi}^a\psi^a + eA_\mu(J^\mu + lG^\mu) + \frac{1}{2}g^2(\bar{\psi}^a\psi^a)^2, \quad (1)$$

where

$$J^\mu = \bar{\psi}^a\gamma^\mu\psi^a, \quad G^\mu = \epsilon^{\mu\nu\rho}\partial_\nu J_\rho,$$

index  $a$  denotes the fermion flavour running from 1 to  $N$  and  $\kappa$ ,  $l$  and  $g$  are coupling constants. Here, as in Ref.[6], we introduce the topological current  $G_\mu$ , associated with the electromagnetic current  $J_\rho$ , which describes the induced charge and current density. We choose  $\gamma$ -matrices to be

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2. \quad (2)$$

The equations of motion are

$$\partial_\nu F^{\nu\mu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} = -e(J^\mu + lG^\mu), \quad (3)$$

$$\gamma^\mu(i\partial_\mu + eA_\mu)\psi^a - m\psi^a + g^2\sum_b(\bar{\psi}^b\psi^b)\psi^a - e\epsilon^{\mu\nu\rho}(\partial_\nu A_\mu)\gamma_\rho\psi^a = 0. \quad (4)$$

We choose the gauge  $A_0 = 0$  and consider the gauge field  $A_i$  to be static. By taking the fermion field  $\psi$  in component form,  $\psi^a = \begin{pmatrix} \psi_+^a \\ \psi_-^a \end{pmatrix} e^{-iE_f t}$ , the equations of motion (4) can be written as the coupled equations for  $\psi_+^a$  and  $\psi_-^a$ ;

$$\begin{aligned} [E_f - m + g^2 \sum_b (|\psi_+^b|^2 - |\psi_-^b|^2) - el\epsilon^{ij0} \partial_j A_i] \psi_+^a &= D_- \psi_-^a - it[(\partial_1 - i\partial_2)E_f] \psi_-^a, \\ [-E_f - m + g^2 \sum_b (|\psi_+^b|^2 - |\psi_-^b|^2) + el\epsilon^{ij0} \partial_j A_i] \psi_-^a &= D_+ \psi_+^a - it[(\partial_1 + i\partial_2)E_f] \psi_+^a, \end{aligned} \quad (5)$$

where  $D_\pm = D_1 \pm iD_2$ ,  $D_i = \partial_i - ieA_i$ . If we let

$$\psi^a = \psi_+^a \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iE_f t}, \quad (6)$$

the equations of motion (5) reduce to

$$\begin{aligned} (E_f - m + g^2 \rho_+ - el\epsilon^{ij0} \partial_j A_i) \psi_+^a &= 0, \\ D_+ \psi_+^a - it[(\partial_1 - i\partial_2)E_f] \psi_+^a &= 0, \end{aligned} \quad (7)$$

and if we take

$$\psi^a = \psi_-^a \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iE_f t}, \quad (8)$$

Eq.(5) becomes

$$\begin{aligned} (E_f + m + g^2 \rho_- - el\epsilon^{ij0} \partial_j A_i) \psi_-^a &= 0, \\ D_- \psi_-^a - it[(\partial_1 + i\partial_2)E_f] \psi_-^a &= 0, \end{aligned} \quad (9)$$

where  $J_\pm^0 = \rho_\pm = \sum_a |\psi_\pm^a|^2$ . The Eqs. (7) and (9) show that the fermion fields,  $\psi_+$  and  $\psi_-$ , satisfy the self-duality conditions.

From the form of the solutions (6) or (8), we find

$$J^i = \bar{\psi}^a \gamma^i \psi^a = 0, \quad (10)$$

which implies that the induced charge also vanishes;

$$G^0 = \epsilon^{0ij} \partial_i J_j = 0. \quad (11)$$

The magnetic fields, for each case of (6) and (8) , then reduce to

$$B_+ = -F_{12} = \frac{e}{\kappa}\rho_+ \quad or \quad B_- = -F_{12} = \frac{e}{\kappa}\rho_- \quad (12)$$

respectively.

Written in (2+1)-dimensional notation the topological current takes the form

$$G^i = \epsilon^{ij0}\partial_j J_0, \quad (13)$$

which is related to the induced current [6] by,

$$G_{ind}^i = l\epsilon^{ij0}\partial_j J_0. \quad (14)$$

As discussed in Ref.[6], this induced current comes from the magnetic dipole moment density,

$$\vec{m} = \frac{\mu}{Q}J_0\hat{z} = lJ_0\hat{z}, \quad (15)$$

through the relation  $\vec{G}_{ind} = \vec{\nabla} \times \vec{m}$ . Here, the magnetic dipole moment is  $\mu\hat{z}$  ( $\hat{z}$  is a unit vector perpendicular to the plane in consideration),  $Q$  is a total charge, and  $J_0$  is a charge density. We note that the field equation (3) can then be written as

$$\partial_j F^{ji} = -el\epsilon^{ij0}\partial_j J_0. \quad (16)$$

By using Eqs. (12) and (16), we find the constant  $l$  to be  $l = -\frac{1}{\kappa}$ . We thus find that there exists the magnetic dipole moment,  $\mu = -\frac{Q}{\kappa}$ , in the system.

By using Eqs. (7) and (9), the Hamiltonian density  $\mathcal{H}$  reduces to

$$\mathcal{H} = \frac{1}{2}F_{12}^2 \pm m\rho_{\pm} - \frac{1}{2}g^2\rho_{\pm}^2 + \frac{e^2 l}{\kappa}\rho_{\pm}^2, \quad (17)$$

where the  $\pm$  signs are for the solutions of the forms (6) and (8), respectively. The total energy of the system can be written as

$$\begin{aligned} E &= \int d^2r \mathcal{H} = \int d^2r \left[ \frac{1}{2}F_{12}^2 \pm m\rho_{\pm} - \frac{1}{2}g^2\rho_{\pm}^2 + \frac{e^2 l}{\kappa}\rho_{\pm}^2 \right] \\ &= \int d^2r \left[ \pm m\rho_{\pm} + \frac{1}{2}\left(\frac{e^2}{\kappa^2} + \frac{2e^2 l}{\kappa} - g^2\right)\rho_{\pm}^2 \right]. \\ &= \int d^2r \left[ \pm m\rho_{\pm} - \frac{1}{2}\left(\frac{e^2}{\kappa^2} + g^2\right)\rho_{\pm}^2 \right]. \end{aligned} \quad (18)$$

The general solutions for the self-duality equation (7) and (9) are well-known[8]. To solve the self-dual equations, we note that when  $\psi_{\pm}$  are decomposed into its phase and amplitude,

$$\psi_{\pm} = \sqrt{\rho_{\pm}} e^{i\omega_{\pm}}, \quad (19)$$

where  $\psi_{\pm} = \sum_a \psi_{\pm}^a$ , the self-dual equations (7) and (9) can be written as

$$e \nabla \times \vec{A} = \pm \nabla \times (\nabla \omega_{\pm}) \mp \nabla \times (\nabla E_f) t \mp \frac{1}{2} \nabla^2 \ln \rho_{\pm}. \quad (20)$$

From Eqs. (12) and (20) we obtain the equation for the charge density  $\rho_{\pm}$ :

$$\nabla^2 \ln \rho_{\pm} = \mp \frac{2e^2}{\kappa} \rho_{\pm}. \quad (21)$$

Eq.(21) is the Liouville equation which is completely integrable. When we take the solution (6),  $\kappa > 0$  is required in order to have nonsingular positive charge density  $\rho_+$ . If we take the solution (8), however,  $\kappa < 0$  is required for the nonsingular charge density  $\rho_-$ . That is, both solutions involve only one of the (2+1) dimensional spinor field components, depending on the sign of  $\kappa$ . Therefore, the most general circularly symmetric nonsingular solutions to the Liouville equations involve two positive constants  $r_{\pm}$  and  $\mathcal{N}_{\pm}$  [3]:

$$\rho_{\pm} = \pm \frac{4\kappa \mathcal{N}_{\pm}^2}{e^2 r^2} \left[ \left( \frac{r_{\pm}}{r} \right)^{\mathcal{N}_{\pm}} + \left( \frac{r}{r_{\pm}} \right)^{\mathcal{N}_{\pm}} \right]^{-2}. \quad (22)$$

where  $r_{\pm}$  are scale parameters and the (+) sign is for the positive  $\kappa$  and the (−) sign for the negative  $\kappa$ . To fix  $\mathcal{N}_{\pm}$ , we observe that regularity at the origin,  $\rho_{\pm} \xrightarrow{r \rightarrow 0} r^{2\mathcal{N}_{\pm}-2}$ , and at infinity,  $\rho_{\pm} \xrightarrow{r \rightarrow \infty} r^{-2\mathcal{N}_{\pm}-2}$ , require  $\mathcal{N}_{\pm} \geq 1$ . Especially, for single-valued  $\psi_{\pm}$ ,  $\mathcal{N}_{\pm}$  must be a positive integer. The charge density  $\rho_{\pm}$  are given by the time-component of  $J^{\mu}$  (here,  $G^0 = 0$ ). By using Eq.(22) the total charge is given by

$$Q_{\pm} = \int \rho_{\pm} d^2 r = \pm \frac{2\kappa \mathcal{N}_{\pm}}{e^2} > 0, \quad (23)$$

which implies that the parameter  $\mathcal{N}_{\pm}$  describes the total charge of the system.

From the solution (22), the total energy of the system is shown to be

$$E = \pm m \frac{2\kappa\mathcal{N}_{\pm}}{e^2} - \left[ \left( \frac{e^2}{\kappa^2} + g^2 \right) \left( \frac{4\pi\kappa^2\mathcal{N}_{\pm}^3\Gamma(2 - \frac{1}{\mathcal{N}_{\pm}})\Gamma(2 + \frac{1}{\mathcal{N}_{\pm}})}{3e^4r_{\pm}^2} \right) \right]. \quad (24)$$

The total energy (24) satisfies the relation

$$E(\mathcal{N}_1) + E(\mathcal{N}_2) + \cdots + E(\mathcal{N}_i) \geq E(\mathcal{N}) \quad (25)$$

where  $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2 + \cdots + \mathcal{N}_i$ . Unlike other Chern-Simons solitons, therefore,  $\mathcal{N}$ -soliton solutions of this model (1) have binding energies. In other words, it is energetically more stable to become  $\mathcal{N}$ -soliton than to be in a state of  $\mathcal{N}$  separate single ( $\mathcal{N} = 1$ ) solitons.

Because the vortex solutions in this theory are nontopological, their stability is not guaranteed automatically [2] [9]. By using Eq.(10) and (11), however, we see that the electromagnetic current  $J^{\mu}$  and the topological current  $G^{\mu}$  are conserved;

$$\partial_{\mu}J^{\mu} = 0 \quad and \quad \partial_{\mu}G^{\mu} = 0. \quad (26)$$

This implies that  $Q_{\pm}$  and  $\mathcal{N}_{\pm}$  are conserved; i.e.

$$\dot{Q}_{\pm} = \dot{\mathcal{N}}_{\pm} = 0. \quad (27)$$

Thus the stability of the  $\mathcal{N}_{\pm}$ -soliton solutions is guaranteed by the charge conservation laws of the system.

We note that, in the limit  $l \rightarrow 0$ ,  $g \rightarrow 0$ ,  $e^2 \rightarrow \infty$  and  $\frac{\kappa}{e^2} \rightarrow 1$ , the Lagrangian (1) reduces to that of Li and Bhaduri [4]. It is easy to show that the solutions and the total energy (24) also reduce to that of Li and Bhaduri in this limit. We finally note that the four fermion interaction term does not affect the structure of the soliton solutions. In other words, the soliton solutions exist even in the limit  $g \rightarrow 0$ , for which case the total energy is given by (24) with  $g = 0$ .

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